# Metaheuristic approaches for MWT and MWPT problems

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**Abstract.** It is known that the Minimum Weight Triangulation problem is NP-hard. Also the complexity of the Minimum Weight Pseudo-Triangulation problem is unknown, yet it is suspected to be also NP-hard. Therefore we focused on the development of approximate algorithms to find high quality triangulations and pseudo-triangulations of minimum weight. In this work we propose two metaheuristics to solve these problems: Ant Colony Optimization (ACO) and Simulated Annealing (SA). For the experimental study we have created a set of instances for MWT and MWPT problems, since no reference to benchmarks for these problems were found in the literature. Through experimental evaluation, we assess the applicability of the ACO and SA metaheuristics for MWT and MWPT problems. These results are compared with those obtained from the application of deterministic algorithms for the same problems (Delaunay Triangulation for MWT and a Greedy algorithm respectively for MWT and MWPT).

## Introduction

In Computational Geometry there are many optimization problems that are either NP-hard or such that no polynomial algorithms are known to solve them. Examples of these optimization problems are those related to special geometric configurations, such as *triangulations* and *pseudo-triangulations*, which are interesting to investigate due to their use in many fields of application.

Minimizing the total length has been one of the main optimality criteria for triangulations and pseudo-triangulations. Indeed, the Minimum Weight Triangulation (MWT) and the Minimum Weight Pseudo-Triangulation (MWPT) minimize the sum of edge lengths, providing a quality measure for determining how good is a structure. The complexity of computing a minimum weight triangulation has been one of the most longstanding open problems in Computational Geometry, introduced by Garey and Johnson in their open problems list, and various approximation algorithms were proposed over time. It has been the subject of numerous investigations for identifying criteria to include certain edges of the MWT. The paper [**2**] presents the implementations of the LMT heuristic for computing MWT problems, that can compute the "exact" MWT for well-behaved point sets very fast. It is also worth noticing that, to our best knowledge, there are no reports about demonstration of such results. Aichholzer et al. [**1**] disprove the conjecture that the LMT-skeleton coincides with the intersection of all locally minimal triangulations, LMT($P$), being $P$ a polygon in the plane, even for a convex polygon. Later, Mulzer and Rote proved in 2006 that MWT construction is an NP-hard problem [**12**].



CRM Documents, vol. **8**, Centre de Recerca Matemàtica, Bellaterra (Barcelona), 2011

The complexity of the MWPT problem is unknown. However, Levcopoulos and Gudmundsson [**8**] show that a 12-approximation of an MWPT can be computed in $O(n^3)$ time. They give an $O(\log n \cdot w(MST))$ approximation of an MWPT, in $O(n \log n)$ time, where $w(MST)$ is the weight of the minimum Euclidean spanning tree, which is a subset of the obtained structure.

Given the inherent difficulty of the above mentioned problems, the approximate algorithms arise as alternative candidates for MWT and MWPT problems. These algorithms can obtain approximate solutions to the optimal solutions, and they can be specific for a particular problem or they can be part of a general applicable strategy in the resolution of different problems. The metaheuristic methods satisfy these properties. These algorithms have a simple implementation and they can efficiently find good solutions for NP-hard optimization problems [**11**]. In this work we use the *Ant Colony Optimization* (ACO) [**4**] and *Simulated Annealing* (SA) [**3, 9, 10**] metaheuristics, and we compare them with Delaunay Triangulation and Greedy algorithms for triangulations and pseudo-triangulations. Previous works about approximations on MWT and MWPT problems using the ACO and SA metaheuristics, were presented in [**5, 7**]. It is also worth noticing that, to the best knowledge of the authors, there are no reports in the literature of extensive experimental evaluations using exact algorithms or metaheuristic techniques.

This paper is organized as follows. In the next section, we review some theoretical aspects of ACO and SA metaheuristics applied to MWT and MWPT problems. Section 2 resumes the experimental evaluation. The last section is devoted to conclusions and future works.

# 1 ACO and SA metaheuristics applied to MWT and MWPT problems

The ACO metaheuristic involves a family of algorithms in which a colony of artificial ants cooperate in finding good solutions to difficult discrete optimization problems [**4**]. An artificial ant in an ACO algorithm is a stochastic constructive procedure that incrementally builds a solution by adding opportunely defined solution components to a partial solution under construction. Therefore, the ACO metaheuristic can be applied to any combinatorial optimization problem for which a constructive graph can be defined. For details of designs, implementations, and parameter settings of the ACO-MWT and ACO-MWPT algorithms, see [**5, 7**].

The SA metaheuristic tries to minimize the limitation of the local search algorithms, which stops as soon as they find a local extreme. For that, it is allowed to accept solutions of worse quality than the current solution with a certain probability. This probability depends on a parameter $T$, called temperature, which decreases over the algorithm iterations according to a decrement rule [**3**]. In regards of the design of the algorithms, the parameter settings and the details of implementations for SA-MWT and SA-MWPT algorithms; see [**6**]. SA2P-MWPT is an improved version of SA-MWPT, which involves a double pass under certain criteria, considering the best results obtained in some temperatures.

## 2 Experimental evaluation

To the best knowledge of the authors, there do not exist collections of instances in the literature for MWT and MWPT problems. Consequently, no benchmarking data are publicly available that allow us to compare our proposal with some other algorithm previously studied. According to that, we design an *instance generator*. Therefore, we have generated respectively a collection of 10 instances of size 40/80/120/160/200; i.e., a total of 50 problem instances. Each instance is called LD$n$-$i$ where $n$ denotes the size of the $i$-instance, with $1 \leq i \leq 10$. The instance generator uses different functions of the CGAL Library. The points are randomly generated, uniformly distributed, and, for each point $(x, y)$, the coordinates $x$, $y$ are in $[0, 1000]$. For the purpose of this work, we assume that there are non-collinear points. The proposed algorithms were implemented in C language and run on a BACO parallel cluster.

In Table 1 we show the results for ACO-MWT, SA-MWT, Greedy Triangulation (GT), and Delaunay Triangulation (DT). The best results were obtained with the SA-MWT algorithm using local retriangulation neighborhood and with different temperature decrement rules (*FSA*, *VFSA*, and geometric decrement with $\alpha = 0.8$).

Especially for pseudo-triangulations, the Greedy Pseudo-Triangulation (GPT) algorithm builds a pseudo-triangulation starting with one face. This face has the edges obtained by the convex hull of the point set $P$, i.e., $CH(P)$. For the solution construction, the $P$ set is partitioned in faces. This process finishes when all faces are pseudo-triangles without interior points. A face is divided in two faces when there are interior points, or is not a pseudo-triangle. Thus, the partition can be done if $i$) there are at least one interior point and two points in the border; or $ii$) there is no interior point, so two points located on the border are chosen. Such selection is performed by selecting those points that generate the edges lead to local minimum weight.

| | *ACO-MWT* | *SA-MWT* | *GT* | *DT* |
|---|---|---|---|---|
| LD40-1 | 5493047 | **5463745** | 5477181 | 5666348 |
| LD40-2 | 4661242 | **4659552** | **4659552** | 4722381 |
| LD40-3 | 5502567 | **5478923** | 5489487 | 5663032 |
| LD40-4 | **5745772** | **5745772** | 5751867 | 6289829 |
| LD80-1 | 6242505 | **6220029** | 6231682 | 6462038 |
| LD80-2 | 7605383 | **7581868** | **7581868** | 8081573 |
| LD80-3 | 5836037 | **5828344** | 5845506 | 6143637 |
| LD80-4 | 6217040 | **6147234** | **6147234** | 6460311 |

TABLE 1. MWT: The best (smallest) weights obtained with the mentioned algorithms for sets of 40 and 80 points.

Table 2 shows the results according to the smallest weights obtained using the ACO-MWPT, SA-MWPT, SA2P-MWPT and GPT algorithms. The best results for the SA-MWPT and SA2P-MWPT algorithms were obtained with edge-flip neighborhood and with the *FSA* temperature decrement rule.

ACO algorithms were first used to show that the results from Greedy algorithms and Delaunay Triangulation can be improved. Other preliminary results seem to indicate that the SA and SA2P algorithms are more effective techniques for MWT and MWPT. For more details or an extended version of this work, please refer to the authors.

| | *ACO-MWPT* | *SA-MWPT* | *SA2P-MWPT* | *GPT* |
|---|---|---|---|---|
| LD40-1 | 6115636 | 5817042 | **4181914** | 5312131 |
| LD40-2 | 4442710 | 4778701 | **3384958** | 4292347 |
| LD40-3 | 5684342 | 6391410 | **4169242** | 5794018 |
| LD40-4 | 5627098 | 5575409 | **5047483** | 6245196 |

Table 2. MWPT: The best (smallest) weights obtained with the mentioned algorithms for sets of 40 points.

# 3 Conclusions

In this work we present the design of approximate algorithms for solving the Minimum Weight Triangulation and Minimum Weight Pseudo-Triangulation problems. Another contribution of this research was the creation of a set of instances for the experimental evaluation, as there are no available instances with special properties for building triangulations and pseudo-triangulations. From this initial experimental phase we obtained preliminary results that will guide future experimentation. Actually, we are in the phase of applying a more methodological approach for the experimental design and running the set of instances for all sets of points mentioned. Since the metaheuristics have proven to behave very well in solving this class of NP-hard problems, there are several directions for further research. We intend to use different parameterizations of the ACO and SA algorithms to adapt and implement other metaheuristics, and to develop hybrid metaheuristics to solve the proposed problems.

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